(mostly the) Final Exam "Discrete Mathematics" - 2023

Firstname	
Surname	

Ex 1	Ex 2	Ex 3	Ex 4	Ex 5	Ex 6	Ex 7	Ex 8	Ex 9	Ex 10

Remarks:

- Fill in your name clearly and sign the exam
- You are not allowed to open the exam before the exam starts
- Time allocated to each student is **3 hours = 180 minutes**
- If any unauthorized electronics or preparation material is found with the student, it will lead to disqualification

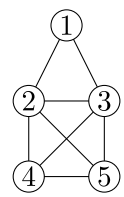
Signature: ______________________________

Exercise 1. Let $2 \leq d \leq n-1$. Compute the number of labeled trees with n vertices such that each vertex has degree d or 1.

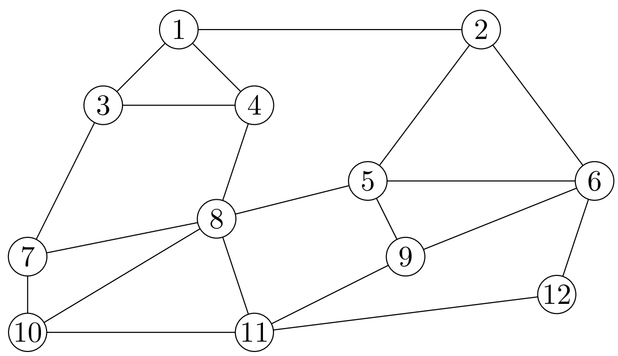
Solution.

Exercise 2. In this exercise we want to "draw graphs in one line without drawing an edge twice". This means finding a path in the graph that contains every edge exactly once. The first vertex of the path does not necessarily have to be the same as its last vertex.

(1) Assuming you want to find such a path in the following graph, at which vertices can you start? Justify your answer.



(2) Can you find such a path in the graph below? Justify your answer.



Solution.

Exercise 3. Prove the inequalities

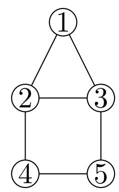
$$\frac{2^n}{n+1} \leq \binom{n}{\lfloor n/2 \rfloor} \leq 2^{n-1}.$$

Solution.

Exercise 4. Consider the graph G shown below. Let $n \in \mathbb{Z}_{\geq 0}$ and let $A(n)$ be the number of closed paths of length n in G, setting $A(0) = 5$.

(1) Show that the sequence $(A(n))_{n=0}^{\infty}$ satisfies a linear recursion.
(2) Find the formula with its initializing values for $A(n)$.

Facts from linear algebra: The trace of a matrix is the sum of its eigenvalues and the eigenvalues are the zeros of the characteristic polynomial. Note that there is no need to compute the eigenvalues.



Solution.

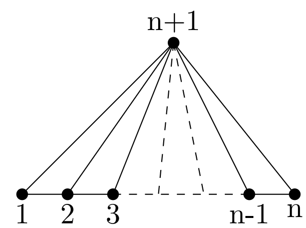
Exercise 5. We call a permutation $(x_1, \ldots, x_{2n})$ of the numbers $1, \ldots, 2n$ pleasant if $|x_i - x_{i+1}| = n$ for at least one $i \in \{1, \ldots, 2n-1\}$. Prove that more than half of all permutations are pleasant for each positive integer n.
Hint: This is a good time to recall the inclusion-exclusion principle and the inequalities that we can derive from it.

Solution.

Exercise 6. Let F_n be the Fibonacci numbers with $F_1 = F_2 = 1$.

(1) Show that for $n \geq 2$ we have
$$F_{2n} = 2F_{2n-2} + \sum_{i=1}^{n-2} F_{2i} + 1.$$

(2) Let $n \geq 1$. In the picture below you can see the so called fan graph for $n+1$ vertices. Show that it has F_{2n} spanning trees.



Solution.

Exercise 7. Let r_n denote the number of distinct prime factors of the number n. Show that

$$\sum_{d|n} |\mu(d)| = 2^{r_n}.$$

Solution.

Exercise 8. In this exercise we consider permutations to be written as a product of disjoint cycles, for example (132)(45).

Assume that the probability is uniform on S_n. Compute the probability that a random permutation in S_n is a cycle of length n.

Solution.

Exercise 9. Let $R(r, s)$ be the Ramsey numbers. Show that for $r, s \geq 2$ the following inequality holds:

$$R(r, s) \leq R(r-1, s) + R(r, s-1)$$

Solution.

Exercise 10. Assume that 8 friends who studied math together are on a canoe tour. They have 4 canoes (and therefore 2 people per canoe) and 7 days. Every morning they divide themselves into 4 groups of 2 people, one group for each boat.

(1) Is it possible to order them such that after the 7 days every one was with every one in a canoe?
(2) Is it always possible to find a combination like the above if they only think of this problem on day 6, and didn't pay attention to the combinations in the first 5 days (assuming they did not already share a boat twice with the same person)?

Solution.